

HELSON AND SUBDIAGONAL OPERATOR ALGEBRAS

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1. COMMUTATIVE ORIGINS

Henry Helson is known for his work in harmonic analysis, function theory, invariant subspaces and related areas of commutative functional analysis. I don't know the extent to which Henry realized, however, that some of his early work inspired significant developments in noncommutative directions - in the area of non-self adjoint operator algebras. Some of the most definitive results were obtained quite recently. I think he would have been pleased by that - while vigorously disclaiming any credit. But surely credit is due; and in this note I will discuss how his ideas contributed to the noncommutative world of operator algebras.

It was my good fortune to be a graduate student at UCLA in the early 1960s, when the place was buzzing with exciting new ideas that had grown out of the merger of classical function theory and the more abstract theory of commutative Banach algebras as developed by Gelfand, Naimark, Raikov, Silov and others. At the same time, the emerging theory of von Neumann algebras and C^* -algebras was undergoing rapid and exciting development of its own. One of the directions of that noncommutative development - though it went unrecognized for many years - was the role of ergodic theory in the structure of von Neumann algebras that was pioneered by Henry Dye [Dye59], [Dye63]. That Henry would become my thesis advisor. I won't say more about the remarkable development of noncommutative ergodic theory that is evolving even today since it is peripheral to what I want to say here. I do want to describe the development of a class of non-self-adjoint operator algebras that relates to analytic function theory, prediction theory and invariant subspaces: Subdiagonal operator algebras.

It is rare to run across a reference to Norbert Wiener's book on prediction theory [Wie57] in the mathematical literature. That may be partly because the book is directed toward an engineering audience, and partly because it was buried as a classified document during the war years. Like all of Wiener's books, it is remarkable and fascinating, but not an easy read for students. It was inspirational for me, and was the source from which I had learned the rudiments of prediction theory that I brought with me to UCLA as a graduate student. Wiener was my first mathematical hero.

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Dirichlet algebras are a broad class of function algebras that originated in efforts to understand the *disk algebra* $A \subseteq C(\mathbb{T})$ of continuous complex-valued functions on the unit circle whose negative Fourier coefficients vanish. Several paths through harmonic analysis or complex function theory or prediction theory lead naturally to this function algebra. I remind the reader that a *Dirichlet algebra* is a unital subalgebra $A \subseteq C(X)$ (X being a compact Hausdorff space) with the property that $A + A^* = \{f + \bar{g} : f, g \in A\}$ is sup-norm-dense in $C(X)$; equivalently, the real parts of the functions in A are dense in the space of real valued continuous functions. One cannot overestimate the influence of the two papers of Helson and Lowdenslager ([HL58], [HL61]) in abstract function theory and especially Dirichlet algebras. Their main results are beautifully summarized in Chapter 4 of Ken Hoffman's book [Hof62].

Along with a given Dirichlet algebra $A \subseteq C(X)$, one is frequently presented with a distinguished complex homomorphism

$$\phi : A \rightarrow \mathbb{C}$$

and because $A + A^*$ is dense in $C(X)$, one finds that there is a unique probability measure μ on X (of course I really mean unique *regular Borel* probability measure) that represents ϕ in the sense that

$$(1.1) \quad \phi(f) = \int_X f d\mu, \quad f \in A.$$

Here we are more concerned with the closely related notion of weak*-Dirichlet algebra $A \subseteq L^\infty(X, \mu)$, in which uniform density of $A + A^*$ in $C(X)$ is weakened to the requirement that $A + A^*$ be dense in $L^\infty(X, \mu)$ relative to the weak*-topology of L^∞ . Of course we continue to require that the linear functional (1.1) should be multiplicative on A .

2. GOING NONCOMMUTATIVE

von Neumann algebras and C^* -algebras of operators on a Hilbert space H are self-adjoint – closed under the $*$ -operation of $\mathcal{B}(H)$. But most operator algebras do not have that symmetry; and for non-self-adjoint algebras, there was little theory and few general principles in the early 1960s beyond the Kadison-Singer paper [KS60] on triangular operator algebras (Ringrose's work on nest algebras was not to appear until several years later).

While trolling the waters for a thesis topic, I was struck by the fact that so much of prediction theory and analytic function theory had been captured by Helson and Lowdenslager, while at the same time I could see diverse examples of operator algebras that seemed to satisfy noncommutative variations of the axioms for weak*-Dirichlet algebras. There had to be a way to put it all together in an appropriate noncommutative context that would retain the essence of prediction theory and contain important examples of operator algebras. I worked on that idea for a year or two and produced a Ph.D. thesis in 1964 – which evolved into a more definitive paper [Arv67].

At the time I wanted to call these algebras *triangular*; but Kadison and Singer had already taken the term for their algebras [KS60]. Instead, these later algebras became known as *subdiagonal* operator algebras.

Here are the axioms for a (concretely acting) subdiagonal algebra of operators in $\mathcal{B}(H)$. It is a pair (\mathcal{A}, ϕ) consisting of a subalgebra \mathcal{A} of $\mathcal{B}(H)$ that contains the identity operator, is closed in the weak*-topology of $\mathcal{B}(H)$, all of which satisfy

SD1: $\mathcal{A} + \mathcal{A}^*$ is weak*-dense in the von Neumann algebra \mathcal{M} it generates.

SD2: ϕ is a conditional expectation, mapping \mathcal{M} onto the von Neumann subalgebra $\mathcal{A} \cap \mathcal{A}^*$.

SD3: $\phi(AB) = \phi(A)\phi(B)$, for all $A, B \in \mathcal{A}$.

What [SD2] means is that ϕ should be an idempotent linear map from \mathcal{M} onto $\mathcal{A} \cap \mathcal{A}^*$, that carries positive operators to positive operators, is continuous with respect to the weak*-topology, and is faithful in the sense that for every positive operator $X \in \mathcal{M}$, $\phi(X) = 0 \implies X = 0$.

We also point out that these axioms differ slightly from the original axioms of [Arv67], but are equivalent when the algebras are weak*-closed.

Examples of subdiagonal algebras:

- (1) The pair (\mathcal{A}, ϕ) , \mathcal{A} being the algebra of all lower triangular $n \times n$ matrices, $\mathcal{A} \cap \mathcal{A}^*$ is the algebra of diagonal matrices, and $\phi : M_n \rightarrow \mathcal{A} \cap \mathcal{A}^*$ is the map that replaces a matrix with its diagonal part.
- (2) Let G be a countable discrete group which can be totally ordered by a relation \leq satisfying $a \leq b \implies xa \leq xb$ for all $x \in G$. There are many such groups, including finitely generated free groups (commutative or noncommutative). Fix such an order \leq on G and let $x \mapsto \ell_x$ be the natural (left regular) unitary representation of G on its intrinsic Hilbert space $\ell^2(G)$, let \mathcal{M} be the weak*-closed linear span of all operators of the form ℓ_x , $x \in G$, and let \mathcal{A} be the weak*-closed linear span of operators of the form ℓ_x , $x \geq e$, e denoting the identity element of G . Finally, let ϕ be the state of M defined by

$$\phi(X) = \langle X\xi, \xi \rangle, \quad X \in M, \quad \xi = \chi_e.$$

If we view ϕ as a conditional expectation from \mathcal{M} to the algebra of scalar multiples of the identity operator by way of $X \mapsto \phi(X)\mathbf{1}$, then we obtain a subdiagonal algebra of operators (\mathcal{A}, ϕ) .

- (3) There are natural examples of subdiagonal algebras in II_1 factors \mathcal{M} that are based on ergodic measure preserving transformations that will be familiar to operator algebraists (see [Arv67]).

In order to formulate the most important connections with function theory and prediction theory, one requires an additional property called *finiteness* in [Arv67]: there should be a distinguished tracial state τ on the von Neumann algebra \mathcal{M} generated by \mathcal{A} that preserves ϕ in the sense that $\tau \circ \phi = \tau$. Perhaps we should indicate the choice of τ by writing $(\mathcal{A}, \phi, \tau)$ rather than (\mathcal{A}, ϕ) , but we shall economize on notation by not doing so.

Recall that the simplest form of *Jensen's inequality* makes the following assertion about functions $f \neq 0$ in the disk algebra: $\log |f|$ is integrable around the unit circle, and the geometric mean of $|f|$ satisfies

$$(2.1) \quad \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) d\theta \right| \leq \exp \frac{1}{2\pi} \int_{\mathbb{T}} \log |f(e^{i\theta})| d\theta.$$

In order to formulate this property for subdiagonal operator algebras we require the determinant function of Fuglede and Kadison [FK52] - defined as follows for invertible operators X in \mathcal{M} :

$$\Delta(X) = \exp \tau(\log |X|),$$

$|X|$ denoting the positive square root of X^*X . There is a natural way to extend the definition of Δ to arbitrary (noninvertible) operators in \mathcal{M} . For example, when \mathcal{M} is the algebra of $n \times n$ complex matrices and τ is the tracial state, $\Delta(X)$ turns out to be the positive n th root of $|\det X|$.

Corresponding to (2.1), we will say that a finite subdiagonal algebra (\mathcal{A}, ϕ) with tracial state τ satisfies *Jensen's inequality* if

$$(2.2) \quad \Delta(\phi(A)) \leq \Delta(A), \quad A \in \mathcal{A},$$

and we say that (\mathcal{A}, ϕ) satisfies *Jensen's formula* if

$$(2.3) \quad \Delta(\phi(A)) = \Delta(A), \quad A \in \mathcal{A} \cap \mathcal{A}^{-1}.$$

It is not hard to show that (2.2) \implies (2.3).

Finally, the connection with prediction theory is made by reformulating a classical theorem of Szegő, one version of which can be stated as follows: For every positive function $w \in L^1(\mathbb{T}, d\theta)$ one has

$$\inf_f \int_{\mathbb{T}} |1 + f|^2 w d\theta = \exp \int_{\mathbb{T}} \log w d\theta,$$

f ranging over trigonometric polynomials of the form $a_1 e^{i\theta} + \dots + a_n e^{in\theta}$. In the noncommutative setting, there is a natural way to extend the definition of determinant to weak*-continuous positive linear functionals ρ on \mathcal{M} , and the proper replacement for Szegő's theorem turns out to be the following somewhat peculiar statement: For every weak*-continuous state ρ on \mathcal{M} ,

$$(2.4) \quad \inf \rho(|D + A|^2) = \Delta(\rho),$$

the infimum taken over $D \in \mathcal{A} \cap \mathcal{A}^*$ and $A \in \mathcal{A}$ with $\phi(A) = 0$ and $\Delta(D) \geq 1$.

In the 1960s, there were several important examples for which I could prove properties (2.2), (2.3) and (2.4); but I was unable to establish them in general. The paper [Arv67] contains the results of that effort. Among other things, it was shown that every subdiagonal algebra is contained in a unique *maximal* one, and that maximal subdiagonal algebras admit *factorization*: *Every invertible positive operator in \mathcal{M} has the form $X = A^*A$ for some $A \in \mathcal{A} \cap \mathcal{A}^{-1}$.* Factorization was then used to show the equivalence of these three properties for arbitrary *maximal* subdiagonal algebras.

3. RESURRECTION AND RESURGENCE

I don't have to say precisely what *maximality* means because, in an important development twenty years later, Ruy Exel [Exe88] showed that the concept is unnecessary by proving the following theorem: *Every (necessarily weak*-closed) subdiagonal algebra is maximal.* Thus, factorization holds *in general* and the three properties (2.2), (2.3), (2.4) are *always* equivalent.

Encouraging as Exel's result was, the theory remained unfinished because no proof existed that Jensen's inequality, for example, was true in general. Twenty more years were to pass before the mystery was lifted. In penetrating work of Louis Labuschagne and David Blecher [Lab05], [BL08], [BL07a], [BL07b] it was shown that, not only are the three desired properties true in general, but virtually all of the classical theory of weak*-Dirichlet function algebras generalizes appropriately to subdiagonal operator algebras.

I hope I have persuaded the reader that there is an evolutionary path from the original ideas of Helson and Lowdenslager, through 40 years of sporadic progress, to a finished and elegant theory of noncommutative operator algebras that embodies a remarkable blend of complex function theory, prediction theory, and invariant subspaces.

REFERENCES

- [Arv67] W. Arveson. Analyticity in operator algebras. *Amer. J. Math.*, 89(3):578–642, July 1967.
- [BL07a] D. Blecher and L. Labuschagne. Noncommutative function theory and unique extensions. *Studia Math.*, 178(2):177–195, 2007. math.OA/0603437.
- [BL07b] D. Blecher and L. Labuschagne. von Neumann algebraic H^p theory. In *Function spaces*, volume 435 of *Contemp. Math.*, pages 89–114. Amer. Math. Soc., 2007. math.OA/0611879.
- [BL08] D. Blecher and L. Labuschagne. A Beurling theorem for noncommutative L^p . *J. Oper. Th.*, 59:29–51, 2008. math.OA/0510358.
- [Dye59] H. A. Dye. On groups of measure preserving transformations, I. *Amer. J. Math.*, 81(1):119–159, 1959.
- [Dye63] H. A. Dye. On groups of measure preserving transformations, II. *Amer. J. Math.*, 85:551–576, 1963.
- [Exe88] R. Exel. Maximal subdiagonal algebras. *Amer. J. Math.*, 110:775–782, 1988.
- [FK52] B. Fuglede and R. V. Kadison. Determinant theory in finite factors. *Ann. Math.*, 55:520–530, 1952.
- [HL58] H. Helson and D. Lowdenslager. Prediction theory and Fourier series in several variables. *Acta Math.*, 99:165–202, 1958.
- [HL61] H. Helson and D. Lowdenslager. Prediction theory and Fourier series in several variables II. *Acta Math.*, 106:175–213, 1961.
- [Hof62] K. Hoffman. *Banach Spaces of Analytic Functions*. Prentice-Hall, Englewood Cliffs, 1962.
- [KS60] R. V. Kadison and I. M. Singer. Triangular operator algebras. *Amer. J. Math.*, 82:227–259, 1960.
- [Lab05] L. Labuschagne. A noncommutative Szegő theorem. *Proc. Amer. Math. Soc.*, 133:3643–3646, 2005.
- [Wie57] N. Wiener. *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*. Technology press of M.I.T. John Wiley and Sons, New York, 1957.